

Inclusions and positive cones of von Neumann algebras

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Abstract

We consider cones in a Hilbert space associated to two von Neumann algebras and determine when one algebra is included in the other. If a cone is associated to a von Neumann algebra, the Jordan structure is naturally recovered from it and we can characterize projections of the given von Neumann algebra with the structure in some special situations.

1 Introduction

The natural positive cone $\mathcal{P}^\natural = \overline{\Delta^{\frac{1}{4}}\mathcal{M}_+\xi_0}$ plays a significant role in the theory of von Neumann algebras (see, for example, [1, 5]) where \mathcal{M} is a von Neumann algebra, ξ_0 is a cyclic separating vector for \mathcal{M} and Δ is the Tomita-Takesaki modular operator associated to ξ_0 . Among them, the result of Connes [6] is of particular interest which characterized the natural positive cones with their geometric properties called selfpolarity, facial homogeneity and orientability, and showed that if two von Neumann algebras \mathcal{M} and \mathcal{N} share a same cone, then there is a central projection q of \mathcal{M} such that $\mathcal{N} = q\mathcal{M} \oplus q^\perp\mathcal{M}'$. Connes used the Lie algebra with an involution of the linear transformation group of \mathcal{P}^\natural in his paper.

In the present paper, instead of \mathcal{P}^\natural , we study $\mathcal{P}^\sharp = \overline{\mathcal{M}_+\xi_0}$, which holds more informations of \mathcal{M} , for example, the subalgebra structure.

In the second section, we study what occurs when $\overline{\mathcal{N}_+\xi_0} \subset \mathcal{P}^\sharp$ where \mathcal{N} is another von Neumann algebra. We consider first the case when ξ_0 is not cyclic for \mathcal{N} and then assume the cyclicity. It turns out that in the latter case \mathcal{N} is included in \mathcal{M} except the part where ξ_0 is tracial.

In the third section, we characterize central projections of \mathcal{M} in terms of \mathcal{P}^\sharp . A projection p is in $\mathcal{M} \cap \mathcal{M}'$ if and only if p and its orthogonal complement p^\perp preserve \mathcal{P}^\sharp .

In the fourth and fifth sections, the Jordan structure on \mathcal{P}^\sharp is studied. We can recover the lattice structure of projections and the operator norm from the order structure of \mathcal{P}^\sharp . Then we can define the square operation on \mathcal{P}^\sharp .

In the final section, using the Jordan structure, a characterization of projections in \mathcal{M} is obtained when the modular automorphism with respect to ξ_0 acts ergodically.

The result of the second section has an easy application to the theory of half-sided modular inclusions [12, 2]. Let $\{U(t)\}$ be a one-parameter group of unitary operators with a generator H which kills ξ_0 . Assume that \mathcal{M} is a factor of type III_1 (or more generally a properly infinite algebra). It is easy to see that $U(t)\mathcal{M}U(t)^* \subset \mathcal{M}$ for $t \geq 0$ if and only if $U(t)$ preserves \mathcal{P}^\sharp for $t \geq 0$. A similar result for \mathcal{P}^\natural and $\{e^{-tH}\}$ has been obtained by Borchers with additional conditions on H [4].

Davidson has obtained conditions for $\{U(t)\}$ to generate a one-parameter semigroup of endomorphisms [7]. The relations with the modular group have been shown to be important in his study.

2 Inclusions of positive cones

Let \mathcal{M} be a von Neumann algebra acting on a Hilbert space \mathcal{H} and ξ_0 be a cyclic separating vector for \mathcal{M} . We denote the modular group by Δ^{it} , the modular conjugation by J , modular automorphism by σ_t and the canonical involution by $S = J\Delta^{\frac{1}{2}}$. The positive cone associated to ξ_0 is denoted by $\mathcal{P}^\sharp = \overline{\mathcal{M}_+\xi_0}$.

Suppose there is another von Neumann algebra \mathcal{N} such that $\overline{\mathcal{N}_+\xi_0} \subset \mathcal{P}^\sharp$. We can define a positive contractive map α from \mathcal{N} into \mathcal{M} as follows.

Lemma 1 *For $a \in \mathcal{N}_+$ there is the unique positive element $\alpha(a) \in \mathcal{M}$ satisfying $a\xi_0 = \alpha(a)\xi_0$. In addition, α is contractive on \mathcal{M}_+ .*

proof By the assumption, we have $a\xi_0 \in \mathcal{P}^\sharp$. Recall that for a vector $a\xi_0$ in \mathcal{P}^\sharp there is a positive linear operator $\alpha(a)$ affiliated to \mathcal{M} such that $a\xi_0 = \alpha(a)\xi_0$ [11].

Since $\|a\|I - a$ is positive, we have $(\|a\|I - a)\xi_0 \in \mathcal{P}^\sharp$. This implies, for every $y \in \mathcal{M}'$,

$$\begin{aligned} \langle \alpha(a)y\xi_0, y\xi_0 \rangle &= \langle \alpha(a)\xi_0, y^*y\xi_0 \rangle \\ &= \langle a\xi_0, y^*y\xi_0 \rangle \\ &\leq \|a\| \langle \xi_0, y^*y\xi_0 \rangle = \|a\| \|y\xi_0\|^2. \end{aligned}$$

Hence $\alpha(a)$ is bounded and in \mathcal{M} . \square

We can easily see that α extends to \mathcal{N} by linearity. Since α is contractive on \mathcal{N}_+ , α is bounded on \mathcal{N}_{sa} .

Lemma 2 *The map α maps every projection to a projection.*

proof Take a projection $e \in \mathcal{N}$. Note that, since α maps \mathcal{N}_+ into \mathcal{M}_+ and is contractive, we have $\alpha(e) \geq \alpha(e)^2$.

Recall that, by the definition of α , we have $\alpha(e)\xi_0 = e\xi_0$. We calculate as follows.

$$\begin{aligned}\langle \alpha(e)^2 \xi_0, \xi_0 \rangle &= \langle \alpha(e)\xi_0, \alpha(e)\xi_0 \rangle \\ &= \langle e\xi_0, e\xi_0 \rangle \\ &= \langle e\xi_0, \xi_0 \rangle \\ &= \langle \alpha(e)\xi_0, \xi_0 \rangle.\end{aligned}$$

This implies that $\langle (\alpha(e) - \alpha(e)^2) \xi_0, \xi_0 \rangle = 0$. As we noted above, $\alpha(e) - \alpha(e)^2$ must be positive, hence the vector $(\alpha(e) - \alpha(e)^2)^{\frac{1}{2}} \xi_0$ must vanish. By the separating property of ξ_0 , we see $\alpha(e) = \alpha(e)^2$. \square

Recall that a linear mapping ϕ which preserves every anticommutator is called a Jordan homomorphism:

$$\phi(xy + yx) = \phi(x)\phi(y) + \phi(y)\phi(x).$$

Now we show the following lemma. The proof of it is essentially taken from [9].

Lemma 3 *The map α is a Jordan homomorphism.*

proof Let e and f be mutually orthogonal projections in \mathcal{N} . Then $e + f$, $\alpha(e)$, $\alpha(f)$ and $\alpha(e) + \alpha(f)$ are projections. We see the range of $\alpha(e)$ and the range of $\alpha(f)$ are mutually orthogonal because if not, then the sum $\alpha(e) + \alpha(f)$ could not be a projection. This implies that

$$\alpha(e)\alpha(f) = \alpha(f)\alpha(e) = 0.$$

In particular, α maps the positive (resp. negative) part of a self-adjoint element x to the positive (reps. negative) part of $\alpha(x)$. From this we see that α is contractive on \mathcal{N}_{sa} .

Next suppose we have commuting projections $e, f \in \mathcal{N}$. Remark that, since $ef \leq e$, positivity of α assures $\alpha(ef) \leq \alpha(e)$. Recalling that in this case ef and e are projections, we see the range of $\alpha(ef)$ is included in the range of $\alpha(e)$. Thus we have $\alpha(ef)\alpha(e) = \alpha(ef)$.

Now noting $e - ef$ and f are mutually orthogonal projections, we have

$$0 = \alpha(e - ef)\alpha(e) = \alpha(e)\alpha(f) - \alpha(ef).$$

Hence α preserves products of commuting projections.

Since every self-adjoint element in a von Neumann algebra is a uniform limit of linear combinations of mutually orthogonal projections, and since α is continuous in norm on \mathcal{N}_{sa} , α preserves products of commuting self-adjoint elements. In particular, α perserves the square of self-adjoint elements.

This implies that, firstly, α preserves Jordan products of self-adjoint elements $ab + ba = (a + b)^2 - a^2 - b^2$. This shows

$$\begin{aligned}\alpha(ab + ba) &= \alpha((a + b)^2) - \alpha(a^2) - \alpha(b^2) \\ &= \alpha(a + b)^2 - \alpha(a)^2 - \alpha(b)^2 \\ &= \alpha(a)\alpha(b) + \alpha(b)\alpha(a).\end{aligned}$$

Secondly, α preserves squares of arbitrary elements $(a + ib)^2 = a^2 + i(ab + ba) - b^2$:

$$\begin{aligned}\alpha((a + ib)^2) &= \alpha(a^2 + i(ab + ba) - b^2) \\ &= \alpha(a^2) + i\alpha(ab + ba) - \alpha(b^2) \\ &= \alpha(a)^2 + i(\alpha(a)\alpha(b) + \alpha(b)\alpha(a)) - \alpha(b)^2 \\ &= (\alpha(a) + i\alpha(b))^2.\end{aligned}$$

Finally, α preserves Jordan products of arbitrary elements $xy + yx = (x + y)^2 - x^2 - y^2$:

$$\begin{aligned}\alpha(xy + yx) &= \alpha((x + y)^2) - \alpha(x^2) - \alpha(y^2) \\ &= \alpha(x + y)^2 - \alpha(x)^2 - \alpha(y)^2 \\ &= \alpha(x)\alpha(y) + \alpha(y)\alpha(x).\end{aligned}$$

This completes the proof. \square

Here we need the following result on Jordan homomorphisms of Jacobson and Rickart [8].

Proposition 4 *Suppose ϕ is a unital Jordan homomorphism from an algebra \mathcal{A} into \mathcal{B} . Suppose further that \mathcal{A} has a system of matrix units. Then there is a central idempotent g of the algebra generated by $\phi(\mathcal{A})$ such that $\phi(\cdot)g$ is homomorphic and $\phi(\cdot)(I - g)$ is antihomomorphic.*

Note that every von Neumann algebra \mathcal{N} decomposes into the commutative part, the I_n parts, the II_1 part, and the properly infinite part. On the first one α causes no problem and on the remaining parts we can apply Proposition 4 to the case in which $\phi = \alpha$, $\mathcal{A} = \mathcal{N}$, $\mathcal{B} = \mathcal{M}$. Examining the proof, we see if ϕ is self-adjoint, then e is an central projection of $\alpha(\mathcal{N})''$ (the argument here is due to Kadison [9]).

Next, we show the normality of α .

Lemma 5 *The map α is a normal linear mapping from \mathcal{N} into \mathcal{M} .*

proof We only have to show that for any normal functional φ on \mathcal{M} the functional $\varphi \circ \alpha$ on \mathcal{N} is normal. Note that, since \mathcal{M} has a separating vector ξ_0 , we may assume $\varphi(\cdot) = \langle \cdot \eta_1, \eta_2 \rangle$ for some $\eta_1, \eta_2 \in \mathcal{H}$.

Recall that a linear functional on a von Neumann algebra is normal if and only if it is continuous on every bounded set in the weak operator topology.

Now suppose that we have a WOT-converging bounded net $x_i \rightarrow x$ in \mathcal{N} . Obviously $\{x_i \xi_0\}$ converges to $x \xi_0$ weakly. By the definition of α , we see $\{\alpha(x_i) \xi_0\}$ converges to $\alpha(x) \xi_0$ weakly. We have, for any $y_1, y_2 \in \mathcal{M}'$,

$$\begin{aligned} \langle \alpha(x_i) y_1 \xi_0, y_2 \xi_0 \rangle &= \langle y_1 \alpha(x_i) \xi_0, y_2 \xi_0 \rangle \\ &= \langle \alpha(x_i) \xi_0, y_1^* y_2 \xi_0 \rangle \\ &\rightarrow \langle \alpha(x) \xi_0, y_1^* y_2 \xi_0 \rangle \\ &= \langle \alpha(x) y_1 \xi_0, y_2 \xi_0 \rangle. \end{aligned}$$

First we assume $\{x_i\}$ is a net of self-adjoint elements. Then for arbitrary $\eta_1, \eta_2 \in \mathcal{H}$ the convergence $\langle \alpha(x_i) \eta_1, \eta_2 \rangle \rightarrow \langle \alpha(x) \eta_1, \eta_2 \rangle$ holds since $\{x_i\}$ is a bounded net, α is contractive on \mathcal{N}_{sa} , and ξ_0 is cyclic for \mathcal{M}' .

Then we can obtain the convergence for arbitrary bounded WOT-converging net $\{x_i\}$ since we have the decomposition

$$x_i = \frac{x_i + x_i^*}{2} + i \frac{x_i - x_i^*}{2i}$$

and each part of the net is self-adjoint or antiself-adjoint, bounded and WOT-converging. \square

We combine this lemma and the proposition of Jacobson and Rickart to get the following.

Lemma 6 *There is a normal homomorphism β and normal antihomomorphism γ of \mathcal{N} into \mathcal{M} such that $\alpha(x) = \beta(x) + \gamma(x)$ and the range of β and γ are mutually orthogonal.*

In addition, there are central projections $e, f \in \mathcal{N}$ and a central projection $g \in \alpha(\mathcal{N})''$ such that $\alpha(e \cdot)g = \beta(\cdot)$ is an isomorphism of $\mathcal{N}e$ and $\alpha(f \cdot)g^\perp = \gamma(\cdot)$ is an antiisomorphism of $\mathcal{N}f$.

proof We know from Proposition 4 that there is a central projection $g \in \alpha(\mathcal{N})''$ such that $\beta(\cdot) = \alpha(\cdot)g$ is a homomorphism of \mathcal{N} and $\gamma(\cdot) = \alpha(\cdot)g^\perp$ is an antihomomorphism of $\mathcal{N}f$. Then just take e as the support of β and f as the support of γ . Since α is normal, so are β and γ and the definitions of e and f are legitimate. \square

Lemma 7 *The von Neumann algebra $\mathcal{N}f$ is finite.*

proof Let $\mathcal{N}h$ be the properly infinite part of $\mathcal{N}f$. We have $g^\perp \alpha(xy) = g^\perp \alpha(y) \alpha(x) = \alpha(y) g^\perp \alpha(x)$ for $x, y \in \mathcal{N}h$.

Again take $x, y \in \mathcal{N}h$. By the definition of α , we have

$$\begin{aligned}
g^\perp xy\xi_0 &= g^\perp \alpha(xy)\xi_0 \\
&= \alpha(y)g^\perp \alpha(x)\xi_0 \\
\langle g^\perp xy\xi_0, \xi_0 \rangle &= \langle \alpha(y)g^\perp \alpha(x)\xi_0, \xi_0 \rangle \\
&= \langle g^\perp \alpha(x)\xi_0, \alpha(y^*)\xi_0 \rangle \\
&= \langle g^\perp x\xi_0, y^*\xi_0 \rangle \\
&= \langle yg^\perp x\xi_0, \xi_0 \rangle.
\end{aligned}$$

Since $\mathcal{N}h$ is properly infinite, there is a sequence of isometries $\{v_n\} \subset \mathcal{N}h$ such that $v_n v_n^* \rightarrow 0$ in SOT-topology (That they are isometries means $v_n^* v_n = h$). Now

$$\begin{aligned}
\langle \gamma(h)\xi_0, \xi_0 \rangle &= \langle g^\perp h\xi_0, \xi_0 \rangle \\
&= \langle g^\perp v_n^* v_n \xi_0, \xi_0 \rangle \\
&= \langle v_n g^\perp v_n^* \xi_0, \xi_0 \rangle \\
&\leq \langle v_n v_n^* \xi_0, \xi_0 \rangle \rightarrow 0.
\end{aligned}$$

But since $\gamma(h)$ is a projection in $\alpha(\mathcal{N})'' \subset \mathcal{M}$ and since ξ_0 is separating for \mathcal{M} , $\gamma(h)$ must be zero. Recalling that h is a subprojection of f and that f is the support of γ , we see that $h = 0$. \square

Theorem 8 *Let \mathcal{M} and \mathcal{N} be von Neumann algebras and ξ_0 is a cyclic separating vector for \mathcal{M} . Suppose $\overline{\mathcal{N}_+ \xi_0} \subset \mathcal{P}^\natural$.*

Then we have two disjoint possibilities:

1. *The von Neumann algebra \mathcal{M} has a subalgebra \mathcal{M}_1 such that $\overline{\mathcal{M}_1 \xi_0} = \overline{\mathcal{N}_+ \xi_0}$.*
2. *For any subalgebra \mathcal{M}_2 of \mathcal{M} , its “sharpened cone” $\overline{\mathcal{M}_2 \xi_0}$ cannot coincide with $\overline{\mathcal{N}_+ \xi_0}$ and \mathcal{N} has a finite ideal \mathcal{N}_1 such that there is a subalgebra of \mathcal{M} which is isomorphic to the direct sum of \mathcal{N}_1 and $\mathcal{N}_1^{\text{opp}}$.*

proof Suppose that e and f defined above are mutually orthogonal. Then let us define $\mathcal{M}_1 = \alpha(\mathcal{N})$. Since we have $ef = 0$, it decomposes as follows.

$$\begin{aligned}
\alpha(\mathcal{N}) &= \alpha(\mathcal{N}[e + e^\perp][f + f^\perp]) \\
&= \alpha(\mathcal{N}[ef^\perp + fe^\perp + e^\perp f^\perp]) \\
&= \beta(\mathcal{N}ef^\perp) + \gamma(\mathcal{N}fe^\perp),
\end{aligned}$$

by noting that $\mathcal{N}e^\perp f^\perp$ is the kernel of α .

Since the range of β and γ are mutually orthogonal, and since e and f are central projections, $\alpha(\mathcal{N})$ is a direct sum of $\beta(\mathcal{N}ef^\perp)$ and $\gamma(\mathcal{N}fe^\perp)$.

Let a be a positive element of \mathcal{N} . Then we have

$$\begin{aligned} a\xi_0 &= \alpha(a)\xi_0 \\ &= \beta(ae)\xi_0 + \gamma(af)\xi_0 \\ &= \beta(aef^\perp)\xi_0 + \gamma(af e^\perp)\xi_0. \end{aligned}$$

Conversely it is easy to see that for $b \in \alpha(\mathcal{N})_+$ there is $a \in \mathcal{N}_+$ such that $\alpha(a) = b$, hence we have $a\xi_0 = b\xi_0$. This completes the proof of the claimed equality $\overline{\mathcal{M}_{1+\xi_0}} = \overline{\mathcal{N} + \xi_0}$.

Next, we assume that $ef \neq 0$. Note that $\mathcal{N}ef$ is noncommutative since by the definition of β and γ the commutative part of \mathcal{N} is left to β . In particular g is a nontrivial central projection in $\alpha(\mathcal{N}ef)''$. By Lemma 7, $\mathcal{N}ef$ is finite. One can easily see that $\alpha(\mathcal{N}ef)''$ is a subalgebra of \mathcal{M} which decomposes into the direct sum of $\beta(\mathcal{N}ef)$ and $\gamma(\mathcal{N}ef)$ where the latter is isomorphic to $(\mathcal{N}ef)^{\text{opp}}$.

What remains to prove is that for any subalgebra \mathcal{M}_2 of \mathcal{M} we cannot have the equality $\overline{(\mathcal{N}ef)_+\xi_0} = \overline{\mathcal{M}_2 + \xi_0}$. To see this impossibility, recall that

$$\overline{\mathcal{M}_+\xi_0} = \{A\xi_0 \mid A \text{ is a closed positive operator affiliated to } \mathcal{M}\},$$

since ξ_0 is a separating vector for \mathcal{M} [11]. Similarly we have

$$\overline{\mathcal{M}_2 + \xi_0} = \{A\xi_0 \mid A \text{ is a closed positive operator affiliated to } \mathcal{M}_2\}.$$

Now suppose $a\xi_0 \in \overline{\mathcal{M}_2 + \xi_0}$ for a positive element a of $\mathcal{N}ef$. By the above remark, we have a positive operator A affiliated to \mathcal{M}_2 such that $a\xi_0 = \alpha(a)\xi_0 = A\xi_0$. Then for $y \in \mathcal{M}'$ we have

$$\alpha(a)y\xi_0 = y\alpha(a)\xi_0 = yA\xi_0 = Ay\xi_0,$$

hence A is bounded and $\alpha(a) = A$. This implies $\alpha(a) \in \mathcal{M}_2$ and $\alpha(\mathcal{N}ef) \subset \mathcal{M}_2$. But by Proposition 4 $\alpha(\mathcal{N}ef)$ generates $\beta(\mathcal{N}ef) \oplus \gamma(\mathcal{N}ef)$. We have $\beta(\mathcal{N}ef) \oplus \gamma(\mathcal{N}ef) \subset \mathcal{M}_2$.

We will show that this leads to a contradiction. By the observation above we see that $\overline{\mathcal{M}_2 + \xi_0}$ contains vectors of the form $ga\xi_0, g^\perp b\xi_0$ where $a, b \in (\mathcal{N}ef)_+$.

Suppose the contrary that $ga\xi_0 \in \overline{(\mathcal{N}ef)_+\xi_0}$. By the argument similar to the above one, there is a self-adjoint positive operator A affiliated to $\mathcal{N}ef$ such that $A\xi_0 = ga\xi_0$. Then $g^\perp A\xi_0 = 0$. Noting that f is the support of γ and that ξ_0 is separating for \mathcal{M} , we see $g^\perp e_A \xi_0 = \gamma(e_A)\xi_0$ cannot vanish for any nontrivial projection e_A of $\mathcal{N}ef$.

There are a spectral projection e_A of A , a positive scalar ϵ and $y \in \mathcal{M}'$ such that $A \geq \epsilon e_A$ and $\langle \gamma(e_A)y\xi_0, y\xi_0 \rangle > 0$. Remark that

$$\begin{aligned} g^\perp(A - \epsilon e_A)\xi_0 &\in \overline{g^\perp(\mathcal{N}ef)_+\xi_0} \\ &\subset \overline{g^\perp(\mathcal{N}ef)_+\xi_0} \\ &= \overline{\gamma(\mathcal{N}ef)_+\xi_0}. \end{aligned}$$

Then we have

$$\begin{aligned}
0 &= \langle yg^\perp A\xi_0, y\xi_0 \rangle \\
&= \langle g^\perp A\xi_0, y^*y\xi_0 \rangle \\
&= \langle g^\perp (A - \epsilon e_A)\xi_0, y^*y\xi_0 \rangle + \langle g^\perp \epsilon e_A\xi_0, y^*y\xi_0 \rangle \\
&\geq \langle g^\perp \epsilon e_A\xi_0, y^*y\xi_0 \rangle \\
&= \langle y\gamma(\epsilon e_A)\xi_0, y\xi_0 \rangle \\
&= \epsilon \langle \gamma(e_A)y\xi_0, y\xi_0 \rangle \\
&> 0.
\end{aligned}$$

This contradiction completes the proof of that $\overline{(\mathcal{N}ef)_+\xi_0} \neq \overline{\mathcal{M}_{2+}\xi_0}$. \square

If we further assume the cyclicity of ξ_0 for \mathcal{N} , we have a stronger result. For the proof of it, we need the following lemma. This can be found, for example in [3], but here we present another simple proof.

Lemma 9 *If $\mathcal{A} \subset \mathcal{B}$ is a proper inclusion of von Neumann algebras on a Hilbert space \mathcal{K} and if ζ is a common cyclic separating vector, then \mathcal{B} cannot be finite.*

proof Suppose the contrary, that \mathcal{B} is finite. Then \mathcal{A} must be finite, too. Hence there is a faithful trace τ on \mathcal{B} . Since ζ is separating for \mathcal{B} , there is a vector η such that $\tau(x) = \langle x\eta, \eta \rangle$ by the Radon-Nikodym type theorem. Since τ is faithful, η must be separating for \mathcal{B} .

We can see that η is cyclic for \mathcal{B} as follows. Denote the orthogonal projection onto $\overline{\mathcal{B}\eta}$ by p . By separation verified above, we have $\overline{\mathcal{B}'\eta} = \mathcal{K}$. On the other hand, by assumption, $\overline{\mathcal{B}\zeta} = \overline{\mathcal{B}'\zeta} = \mathcal{K}$. By the general theory of equivalence of projections, $p \sim I$ in \mathcal{B} . But recalling that \mathcal{B} is finite, we see that $p = I$, i.e., η is cyclic.

By the same reasoning, η is cyclic separating tracial for \mathcal{A} . Then the modular conjugations $J_{\mathcal{A}}$ and $J_{\mathcal{B}}$ with respect to η must coincide and we have the required equation.

$$\mathcal{A}' \supset \mathcal{B}' = J_{\mathcal{B}}\mathcal{B}J_{\mathcal{B}} = J_{\mathcal{A}}\mathcal{B}J_{\mathcal{A}} \supset J_{\mathcal{A}}\mathcal{A}J_{\mathcal{A}} = \mathcal{A}'.$$

This contradicts the assumption that the inclusion $\mathcal{A} \subset \mathcal{B}$ is proper. \square

Theorem 10 *Let \mathcal{M} and \mathcal{N} be von Neumann algebras and ξ_0 be a vector cyclic separating for \mathcal{M} and cyclic for \mathcal{N} . Suppose $\overline{\mathcal{N}_+\xi_0} \subset \mathcal{P}^\sharp$.*

Then we have the following.

1. *The vector ξ_0 is also separating for \mathcal{N} .*
2. *There is a central projection e in \mathcal{N} such that $\mathcal{N}e \subset \mathcal{M}$.*
3. *The vector $e^\perp \xi_0$ is tracial for $\mathcal{N}e^\perp$.*
4. *$J_{e^\perp}\mathcal{N}e^\perp J_{e^\perp} \subset \mathcal{M}$.*

In particular, \mathcal{N} and $\mathcal{N}e \oplus J_{e^\perp}\mathcal{N}e^\perp J_{e^\perp}$ share the same positive cone $\mathcal{P}_{\mathcal{N}}^\sharp$ where $\mathcal{N}e \oplus J_{e^\perp}\mathcal{N}e^\perp J_{e^\perp} \subset \mathcal{M}$.

proof First we show that the induction by g realizes $\beta(\cdot) = g\alpha(\cdot)$. For arbitrary $x, y \in \mathcal{N}$ we have

$$\begin{aligned} gxy\xi_0 &= g\alpha(xy)\xi_0 \\ &= g\alpha(x)\alpha(y)\xi_0 \\ &= g\alpha(x)y\xi_0 \\ &= \alpha(x)gy\xi_0. \end{aligned}$$

Taking it into consideration that ξ_0 is cyclic for \mathcal{N} , we see that $gx = g\alpha(x) = \alpha(x)g$. But, since this holds for arbitrary $x \in \mathcal{N}$, in particular for self-adjoint elements. If $x = x^*$, then we have

$$gx = \alpha(x)g = (g\alpha(x))^* = (gx)^* = xg.$$

Since this equation is linear for x , we see that $g \in \mathcal{N}'$ and $gx = g\alpha(x)$.

Now recall that we have decomposed α into a normal homomorphism β and a normal antihomomorphism γ . We again denote the support of β by e and the support of γ by f .

Let $\mathcal{N}h$ be the properly infinite part. By Lemma 7 the intersection of h and f is trivial. Thus we have

$$ghx\xi_0 = hg\alpha(hx)\xi_0 = h\alpha(hx)\xi_0 = hx\xi_0,$$

for $x \in \mathcal{N}$. Cyclicity of ξ_0 tells us that $gh = h$. Then for $hx \in \mathcal{N}h$ we get that

$$\alpha(hx) = ghx = hx.$$

In other words, α maps identically on $\mathcal{N}h$. In particular, α is decomposed by h , that is, we have

$$h\alpha(h^\perp) = \alpha(h)\alpha(h^\perp) = 0,$$

since α maps orthogonal projections to orthogonal projections.

Note that $h\xi_0$ is cyclic for $\mathcal{N}h$ since ξ_0 is cyclic for \mathcal{N} . The vector $h\xi_0$ is also separating for $\mathcal{N}h$ since

$$\mathcal{N}h = \alpha(\mathcal{N}h) \subset \mathcal{M}$$

and ξ_0 is separating for \mathcal{M} .

For the proof of remaining part of the theorem, we may assume \mathcal{N} is finite.

Recall that g^\perp commutes with \mathcal{N} . Take $x, y \in \mathcal{N}$ and let us calculate

$$\begin{aligned} \langle xyg^\perp\xi_0, g^\perp\xi_0 \rangle &= \langle g^\perp y\xi_0, g^\perp x^*\xi_0 \rangle \\ &= \langle g^\perp \alpha(y)\xi_0, g^\perp \alpha(x^*)\xi_0 \rangle \\ &= \langle g^\perp \alpha(x)\alpha(y)\xi_0, g^\perp \xi_0 \rangle \\ &= \langle g^\perp \alpha(yx)\xi_0, g^\perp \xi_0 \rangle \\ &= \langle g^\perp yx\xi_0, g^\perp \xi_0 \rangle \\ &= \langle yxg^\perp\xi_0, g^\perp \xi_0 \rangle \end{aligned}$$

This shows that $g^\perp \xi_0$ is a tracial vector for $\mathcal{N}g^\perp$. By assumption, ξ_0 is cyclic for \mathcal{N} , hence $g^\perp \xi_0$ is cyclic for $\mathcal{N}g^\perp$. In addition, it is also separating as follows. If $xg^\perp \xi_0 = 0$ for some $x \in \mathcal{N}g^\perp$, then for any $y \in \mathcal{N}g^\perp$ we have

$$\begin{aligned} \|xyg^\perp \xi_0\|^2 &= \langle y^* x^* xyg^\perp \xi_0, g^\perp \xi_0 \rangle \\ &= \langle xyy^* x^* g^\perp \xi_0, g^\perp \xi_0 \rangle \\ &\leq \|y\|^2 \langle xx^* g^\perp \xi_0, g^\perp \xi_0 \rangle \\ &= \|y\|^2 \langle x^* x g^\perp \xi_0, g^\perp \xi_0 \rangle \\ &= 0, \end{aligned}$$

then the cyclicity implies the separation by $g^\perp \xi_0$.

Now $\mathcal{N}g^\perp$ has the canonical conjugation J_{g^\perp} defined as (the closure of)

$$J_{g^\perp} : g^\perp \mathcal{H} \ni x\xi_0 \longmapsto x^* \xi_0 \in g^\perp \mathcal{H}.$$

On $\mathcal{N}g^\perp$ we have the canonical antihomomorphism

$$\mathcal{N}g^\perp \ni x \longmapsto J_{g^\perp} x^* J_{g^\perp} \in \mathcal{N}g^\perp.$$

In our situation the composition of the induction by g^\perp and this antihomomorphism coincide with the composition of α and the induction by g^\perp . In fact, for any elements $x, y, z \in \mathcal{N}g^\perp$ we have

$$\begin{aligned} \langle J_{g^\perp} (xg^\perp)^* g^\perp J_{g^\perp} yg^\perp \xi, zg^\perp \xi_0 \rangle &= \langle z^* g^\perp \xi_0, x^* y^* g^\perp \xi_0 \rangle \\ &= \langle yxz^* g^\perp \xi_0, g^\perp \xi_0 \rangle \\ &= \langle z^* yxg^\perp \xi_0, g^\perp \xi_0 \rangle \\ &= \langle g^\perp yx\xi_0, zg^\perp \xi_0 \rangle \\ &= \langle g^\perp \alpha(yx)\xi_0, zg^\perp \xi_0 \rangle \\ &= \langle g^\perp \alpha(x)\alpha(y)\xi_0, zg^\perp \xi_0 \rangle \\ &= \langle g^\perp \alpha(x)y\xi_0, zg^\perp \xi_0 \rangle \\ &= \langle g^\perp \alpha(x)y\xi_0, zg^\perp \xi_0 \rangle. \end{aligned}$$

The cyclicity of $g^\perp \xi_0$ shows that $g^\perp \alpha(x) = J_{g^\perp} (xg^\perp)^* J_{g^\perp}$.

Summing up, we get the following formula for α :

$$\begin{aligned} \alpha(x) &= g\alpha(x) + g^\perp \alpha(x) \\ &= gx + J_{g^\perp} g^\perp x^* J_{g^\perp}. \end{aligned}$$

Note that $g\xi_0$ is cyclic separating for $\mathcal{N}g$. In fact, the cyclicity comes from the assumption of ξ_0 's cyclicity and separating property can be seen by observing

$$\mathcal{N}g = g\alpha(\mathcal{N}) \subset \mathcal{M}$$

and by separating property of ξ_0 for \mathcal{M} .

On the other hand, we have seen that $g^\perp \xi_0$ is cyclic separating for $\mathcal{N}g^\perp$ in the way proving that $g^\perp \xi_0$ is a faithful tracial vector.

The direct sum of $\mathcal{N}g$ and $\mathcal{N}g^\perp$ has a cyclic separating vector ξ_0 . These summands are finite because we are assuming that \mathcal{N} is finite and they are induced part of it. Hence $\mathcal{N}g \oplus \mathcal{N}g^\perp$ is also finite.

Clearly \mathcal{N} is a subalgebra of $\mathcal{N}g \oplus \mathcal{N}g^\perp$. So ξ_0 is separating for \mathcal{N} . This is the first statement of the theorem.

Now we have an inclusion of finite von Neumann algebras

$$\mathcal{N} \subset \mathcal{N}g \oplus \mathcal{N}g^\perp$$

and ξ_0 is a common cyclic separating vector. Then they must coincide by Lemma 9. This happens only if g is a projection of \mathcal{N} from the beginning, i.e, g is a central projection of \mathcal{N} .

Recall that induction by g coincides with the homomorphic part of α . Now we know that g is central. Then the support e of the homomorphic part β must be exactly g .

On the other hand, the intersection $e^\perp f^\perp$ of kernels of the homomorphic part β and the antihomomorphic part γ must be trivial. To see this, take $x \in \mathcal{N}$. We have

$$\begin{aligned} e^\perp f^\perp x \xi_0 &= x e^\perp f^\perp \xi_0 \\ &= x \alpha(e^\perp f^\perp) \xi_0 \\ &= 0. \end{aligned}$$

Since ξ_0 is cyclic for \mathcal{N} , we get that $e^\perp f^\perp = 0$.

Since the induction by e realizes the homomorphic part β of α , for the antihomomorphic part γ it holds

$$\gamma(e) = e^\perp \alpha(e) = \alpha(e) - e \alpha(e) = 0.$$

This implies e must be orthogonal to f , which is the support of γ . As their intersection vanishes, we get $f = I - e$.

Recalling $g = e$, we saw that $e^\perp \xi_0$ is a cyclic separating tracial vector for $\mathcal{N}e^\perp$ and the canonical antiisomorphism with respect to $e^\perp \xi_0$ coincides with $e^\perp \alpha$. Then the proof of all the statements in the theorem is done. \square

3 Recovery of central projections

In the following sections we turn to the study of single von Neumann algebra. Again let \mathcal{M} be a von Neumann algebra and ξ_0 be a cyclic separating vector for \mathcal{M} . By Connes' result, \mathcal{P}^\sharp determines \mathcal{M} up to center.

Here we show that the center is easily recovered from \mathcal{P}^\sharp . Let p be a projection \mathcal{P}^\sharp such that $p\mathcal{P}^\sharp \subset \mathcal{P}$ and $p^\perp \mathcal{P}^\sharp \subset \mathcal{P}^\sharp$.

In this situation, we can define a mapping from \mathcal{M} into \mathcal{M} using p .

Lemma 11 *For every $a \in \mathcal{M}_+$ there is $\alpha(a) \in \mathcal{M}_+$ such that $pa\xi_0 = \alpha(a)\xi_0$.*

proof As in the proof of Lemma 1, we have a positive operator $\alpha(a)$ affiliated to \mathcal{M} such that $pa\xi_0 = \alpha(a)\xi_0$ since $pa\xi_0$ is a vector of the positive cone \mathcal{P}^\sharp . This is again bounded for a different reason. In fact, for $y \in \mathcal{M}'$ we have

$$\begin{aligned} \langle \alpha(a)y\xi_0, y\xi_0 \rangle &= \langle \alpha(a)\xi_0, y^*y\xi_0 \rangle \\ &= \langle pa\xi_0, y^*y\xi_0 \rangle \\ &\leq \langle pa\xi_0, y^*y\xi_0 \rangle + \langle p^\perp a\xi_0, y^*y\xi_0 \rangle \\ &= \langle a\xi_0, y^*y\xi_0 \rangle \\ &= \langle ay\xi_0, y\xi_0 \rangle \\ &\leq \|a\| \|y\xi_0\|^2, \end{aligned}$$

where we have used the assumption that p^\perp preserves \mathcal{P}^\sharp . \square

From this we see that $\alpha(a) \leq a$ as self-adjoint operators. The map α extends to a linear mapping of \mathcal{M} .

Lemma 12 *The map α maps every projection to a projection.*

proof Let e be a projection of \mathcal{M} . By the observation above, we have $\alpha(e) \leq e$. Then using the fact $e\alpha(e) = \alpha(e)$ we can calculate

$$\begin{aligned} \langle \alpha(e)^2 \xi_0, \xi_0 \rangle &= \langle \alpha(e)\xi_0, \alpha(e)\xi_0 \rangle \\ &= \langle pe\xi_0, pe\xi_0 \rangle \\ &= \langle pe\xi_0, e\xi_0 \rangle \\ &= \langle \alpha(e), e\xi_0 \rangle \\ &= \langle \alpha(e), \xi_0 \rangle. \end{aligned}$$

We can see that $\alpha(e)^2 = \alpha(e)$ as in the proof of Lemma 2. \square

Then the mapping α is a normal Jordan homomorphism and there is a central projection g of $\alpha(\mathcal{M})'' \subset \mathcal{M}$ such that $\alpha(\cdot)g$ is homomorphic and $\alpha(\cdot)g^\perp$ is antihomomorphic. The proof is similar to the one for the case of subcones.

Now we have the following.

Theorem 13 *Let \mathcal{M} be a von Neumann algebra acting on a Hilbert space \mathcal{H} , ξ_0 be a cyclic separating vector for \mathcal{M} and $\mathcal{P}^\sharp = \overline{\mathcal{M}_+\xi_0}$. Then a projection $p \in \mathcal{B}(\mathcal{H})$ is a central projection of \mathcal{M} if and only if p and p^\perp preserve \mathcal{P}^\sharp .*

proof The “only if” part is trivial.

Let p be a projection which and whose orthogonal complement preserve \mathcal{P}^\sharp . Note that $\alpha(x) \in \mathcal{M}$ and that $\alpha(\alpha(x)) = \alpha(x)$ holds. In fact, we have

$$\alpha(\alpha(x))\xi_0 = p\alpha(x)\xi_0 = pp x\xi_0 = px\xi_0 = \alpha(x)\xi_0,$$

since p is a projection.

As in the situation of subcones, α is a sum of a normal homomorphism and a normal antihomomorphism whose ranges are mutually orthogonal. The kernels of the homomorphism and the antihomomorphism are central projections of \mathcal{M} . Thus the support of α is the orthogonal complement of the intersection of these kernels. In particular it is a central projection $e \in \mathcal{M}$.

Recall that $\alpha(e) \leq e$. Take an arbitrary positive element a from \mathcal{M} . If we apply α to $ea - \alpha(ea)$, since the composition of α and α equals α itself, we have

$$\alpha(ea - \alpha(ea)) = \alpha(ea) - \alpha(ea) = 0.$$

The argument of the left hand side is less than the support of α , hence it must vanish. Thus we see that ea is fixed by α . By linearity, this holds for arbitrary element $x \in \mathcal{M}$ instead of positive element a .

Again since e is the support of α , we have $\alpha(x) = \alpha(xe) = xe$. Comparing this with the definition of α we can determine p .

$$\begin{aligned} px\xi_0 &= \alpha(x)\xi_0 \\ &= ex\xi_0 \end{aligned}$$

With the cyclicity of ξ_0 we see that p equals e . In particular, p must be a central projection of \mathcal{M} . \square

4 Properties of $(\mathcal{P}^\sharp, \xi_0)$

In this section, we study the properties of \mathcal{P}^\sharp coupled with a specified vector ξ_0 . We begin with the following lemma.

Let us write $\zeta \leq \eta$ if $\eta - \zeta \in \mathcal{P}^\sharp$.

Lemma 14 *Let ζ be a vector in \mathcal{P}^\sharp . Then the following hold.*

1. *If $\zeta \leq \xi_0$, then there is a positive contractive operator $a \in \mathcal{M}$ such that $\zeta = a\xi_0$. In this case we say that ζ is contractive.*
2. *If ζ is contractive and if $\zeta \perp (\xi_0 - \zeta)$, then there is a projection $e \in \mathcal{M}$ such that $\zeta = e\xi_0$. When these conditions hold, we call ζ a projective vector.*
3. *If η and ζ are projective and $\zeta \leq \xi_0 - \eta$, then e and f are mutually orthogonal projections where $\eta = e\xi_0$ and $\zeta = f\xi_0$. We say η and ζ are mutually operationally orthogonal.*

proof The proofs of the first and the second statements are same as in the proofs of Lemma 1 and 2 respectively. We do not repeat them here.

Suppose $\eta = e\xi_0$, $\zeta = f\xi_0$ and $\eta \leq \xi_0 - \zeta$. Then according to this order, $e \leq I - f$. When e and f are projections, this shows the mutual orthogonality. \square

We denote the set of contractive vectors by \mathcal{P}_1^\sharp . By the Lemma above, to each vector in \mathcal{P}_1^\sharp there corresponds a positive contractive operator of \mathcal{M} .

Similarly to every vector ζ in $\mathbb{R}_+\mathcal{P}_1^\sharp$ there corresponds a bounded positive operator a of \mathcal{M} . Put $\mathcal{P}_b^\sharp = \mathbb{R}_+\mathcal{P}_1^\sharp$.

Lemma 15 *For an arbitrary vector ζ in \mathcal{P}_b^\sharp there is a least projective vector such that $\eta \geq \zeta$. Let us call η the support of ζ .*

proof As noted above, there is a positive operator a such that $\zeta = a\xi_0$. As we have seen, the order structure of \mathcal{P}_b^\sharp is consistent with this correspondence. Let e be the support projection of a . Then we have $\eta = e\xi_0 \geq a\xi_0 = \zeta$. Hence η is the least projective vector in \mathcal{P}_b^\sharp . \square

Lemma 16 *Every vector ζ in \mathcal{K} is uniquely decomposed as $\zeta = \zeta_+ - \zeta_-$ where ζ_+ and ζ_- are vectors of \mathcal{P}_b^\sharp and supports of ζ_+ and ζ_- are mutually operationally orthogonal.*

proof Since every vector in \mathcal{P}_1^\sharp corresponds to a positive contractive operator in \mathcal{M} , vectors of \mathcal{P}_b^\sharp (resp. \mathcal{K}) correspond to positive operators (resp. self-adjoint operators).

Now the lemma follows from the theory of self-adjoint operators. The self-adjoint operator z corresponding to ζ has the Jordan decomposition $z = z_+ - z_-$ where z_+ and z_- are positive operators of \mathcal{M} whose supports are mutually orthogonal. By Lemma 14, ζ has the corresponding decomposition. \square

Lemma 17 *The cone \mathcal{P}_b^\sharp is dense in \mathcal{P}^\sharp .*

proof For each vector ζ in \mathcal{P}^\sharp there is a positive self-adjoint linear operator A affiliated to \mathcal{M} such that $\zeta = A\xi_0$ [11]. Let E_A be the spectral measure associated to A . Then $AE_A([0, n])$ is bounded positive operator in \mathcal{M} . It is well known that $\{AE_A([0, n])\xi_0\}$ converges to $A\xi_0$. \square

In addition, we can recover the operator norm in terms of \mathcal{P}_b^\sharp . For $\zeta \in \mathcal{P}_b^\sharp$ we define the new “sharp” norm $\|\zeta\|_\sharp$ as follows.

$$\|\zeta\|_\sharp = \sup \left\{ c \geq 0 \mid \frac{1}{c}\zeta \leq \xi_0 \right\}.$$

Lemma 18 *If $a \in \mathcal{M}_+$ and $\zeta = a\xi_0$, then $\|\zeta\|_\sharp = \|a\|$.*

proof We only have to note that $ca\xi_0 \leq \xi_0$ if and only if $ca \leq I$. Then the spectral decomposition of a completes the proof. \square

Put $\mathcal{K} = \mathcal{P}_b^\# - \mathcal{P}_b^\#$. This is a real linear subspace of \mathcal{H} .

To \mathcal{K} we can extend the new norm $\|\cdot\|_\#$ as follows. For $\zeta \in \mathcal{K}$ define

$$\|\zeta\|_\# = \inf \left\{ \max \left\{ \|\zeta_1\|_\#, \|\zeta_2\|_\# \right\} \mid \zeta_1, \zeta_2 \in \mathcal{P}_b^\#, \zeta_1 - \zeta_2 = \zeta \right\}.$$

It is easily seen that if $z \in \mathcal{M}_{sa}$ corresponds to $\zeta \in \mathcal{K}$, we have

$$\max \{ \|z_+\|, \|z_-\| \} = \|z\| = \|\zeta\|_\# = \max \{ \|\zeta_+\|_\#, \|\zeta_-\|_\# \}.$$

5 Jordan structure on $\mathcal{K} + i\mathcal{K}$

First we define the square operation for vectors in \mathcal{K} .

Definition 19 If ζ is a real linear combination of mutually operationally orthogonal projective vectors, i.e. $\zeta = \sum_k c_k \zeta_k$ where $c_k \in \mathbb{R}$ and $\{\zeta_k\}$ are mutually operationally orthogonal, then we define the square of ζ as follows.

$$\zeta^2 = \sum_k c_k^2 \zeta_k.$$

As we have seen in Lemma 14, mutually operationally orthogonal projective vectors $\{\zeta_k\}$ correspond to mutually orthogonal projections $\{e_k\}$. Thus the square of a real linear combination $\sum_k c_k e_k$ equals $\sum_k c_k^2 e_k$ and for these vectors the definition of square is consistent.

The set of vectors which are real linear combinations of mutually operationally orthogonal projective vectors is dense in \mathcal{K} in the sharp norm defined in Section 4. In fact, these vectors correspond to real linear combinations of mutually orthogonal projections in \mathcal{M} , i.e. self-adjoint operators with finite spectra.

Since the sharp norm on \mathcal{K} is consistent with the operator norm on \mathcal{M} , we can extend the definition of square to \mathcal{K} by continuity. We have the following.

$$\text{If } \zeta = z\xi_0 \text{ for } z \in \mathcal{M}_{sa}, \text{ then } \zeta^2 = z^2\xi_0.$$

Once we have defined the square operation on \mathcal{K} , we can define Jordan polynomials as follows. For η and ζ in \mathcal{K} let us define

$$\eta\zeta + \zeta\eta = (\eta + \zeta)^2 - \eta^2 - \zeta^2.$$

Using this, for $\zeta = \zeta_1 + i\zeta_2 \in \mathcal{K} + i\mathcal{K}$ we put

$$\zeta^2 = \zeta_1^2 + i(\zeta_1\zeta_2 + \zeta_2\zeta_1) - \zeta_2^2.$$

As for vectors in \mathcal{K} , we define the ‘‘Jordan product’’ on $\mathcal{K} + i\mathcal{K}$ by

$$\eta\zeta + \zeta\eta = (\eta + \zeta)^2 - \eta^2 - \zeta^2.$$

Using this, finally we define

$$\zeta\eta\zeta = \frac{1}{2} [(\zeta\eta + \eta\zeta)\zeta + \zeta(\zeta\eta + \eta\zeta)] - \frac{1}{2} (\zeta^2\eta + \eta\zeta^2).$$

If $\eta = y\xi_0$ and $\zeta = z\xi_0$ for $y, z \in \mathcal{M}$, then it follows that $\zeta\eta\zeta = zyz\xi_0$. This follows because we have defined square and Jordan polynomials on \mathcal{K} consistently.

If we fix ζ , we give names to the following mappings.

$$\begin{aligned} c_\zeta : \mathcal{K} + i\mathcal{K} \ni \eta &\longmapsto \zeta\eta\zeta \in \mathcal{K} + i\mathcal{K}, \\ \text{od}_\zeta : \mathcal{K} + i\mathcal{K} \ni \eta &\longmapsto \eta - c_\zeta(\eta) - c_{\zeta^\perp}(\eta) \in \mathcal{K} + i\mathcal{K}. \end{aligned}$$

Let $\eta = y\xi_0$ and $\zeta = e\xi_0$ where e is a projection. Then we see that

$$\begin{aligned} c_\zeta(\eta) &= eye\xi_0, \text{ and} \\ \text{od}_\zeta(\eta) &= y\xi_0 - eye\xi_0 - e^\perp ye^\perp \xi_0 = [eye^\perp + e^\perp ye] \xi_0 \end{aligned}$$

correspond to the corner of y and the off-diagonal part of y , respectively.

6 Recovery of projections in \mathcal{M} in the case when $\mathcal{M}^\sigma = \mathbb{C}I$

Let p be a projection of $\mathcal{B}(\mathcal{H})$. We seek a necessary and sufficient condition for p to be a projection of \mathcal{M} .

We need a criterion for a projection in \mathcal{M} to be fixed by the modular automorphism.

Lemma 20 *Let e be a projection in \mathcal{M} . If $px\xi_0 = xe\xi_0$ holds for all $x \in \mathcal{M}$, then we have $e \in \mathcal{M}^\sigma$ and $p = JeJ$.*

proof Note that we get $p\xi_0 = e\xi_0$ if we use the assumption with $x = I$.

Again by the assumption it follows that

$$\begin{aligned} \langle xe\xi_0, \xi_0 \rangle &= \langle px\xi_0, \xi_0 \rangle \\ &= \langle x\xi_0, p\xi_0 \rangle \\ &= \langle x\xi_0, e\xi_0 \rangle \\ &= \langle ex\xi_0, \xi_0 \rangle. \end{aligned}$$

This implies that $e \in \mathcal{M}^\sigma[11]$. In particular, we have

$$e\xi_0 = Se\xi_0 = J\Delta^{\frac{1}{2}}e\xi_0 = Je\xi_0.$$

Now the equality $JeJx\xi_0 = xJeJ\xi_0 = xe\xi_0 = px\xi_0$ and the cyclicity of ξ_0 complete the proof. \square

Recall that $S = J\Delta^{\frac{1}{2}}$ can be defined in terms of $\overline{\mathcal{K}}$ [10].

Theorem 21 *Let p be a projection of \mathcal{M} . There is a projection $e \in \mathcal{M}$ and a central projection $q \in \mathcal{M}$ such that $q^\perp e \in \mathcal{M}^\sigma$ and $p = qe + Jq^\perp e J$ if and only if the following hold:*

1. $p\xi_0 \leq \xi_0$.
2. If $\zeta \leq p\xi_0$, then $p\zeta = \zeta$.
3. If $\zeta \leq p^\perp \xi_0$, then $p^\perp \zeta = \zeta$.
4. For every vector $\xi \in \mathcal{K} + i\mathcal{K}$ we have $p\xi \in \mathcal{K} + i\mathcal{K}$ and
 - (a) $c_{p\xi_0}(p \operatorname{od}_{p\xi_0}(\xi)) = 0$,
 - (b) $c_{p^\perp \xi_0}(p \operatorname{od}_{p\xi_0}(\xi)) = 0$,
 - (c) $(p \operatorname{od}_{p\xi_0}(\xi))^2 = 0$,
 - (d) $(p^\perp \operatorname{od}_{p\xi_0}(\xi))^2 = 0$,
 - (e) $Sp \operatorname{od}_{p\xi_0}(\xi) = p^\perp S \operatorname{od}_{p\xi_0}(\xi)$.

proof First let us show the “only if” part. In this case, we have

$$p\xi_0 = qe\xi_0 + Jq^\perp e J\xi_0 = qe\xi_0 + q^\perp e\xi_0 = e\xi_0 \leq \xi_0,$$

hence the first part of the conditions is satisfied. For the second condition, if $\zeta = z\xi_0 \leq p\xi_0 = e\xi_0$, then the support of z is less than or equal to e and we have

$$p\zeta = qez\xi_0 + zJeq^\perp J\xi_0 = qez\xi_0 + zeq^\perp \xi_0 = z\xi_0 = \zeta.$$

Similar proof works for the third. To see the conditions of the fourth, let $\xi = x\xi_0 \in \mathcal{K} + i\mathcal{K}$. We note that

$$\begin{aligned} c_{p\xi_0}(\xi) &= c_{e\xi_0}(x\xi_0) = exe\xi_0, \\ \operatorname{od}_{p\xi_0}(\xi) &= \operatorname{od}_{e\xi_0}(x\xi_0) = [exe^\perp + e^\perp xe] \xi_0, \\ p \operatorname{od}_{p\xi_0}(\xi) &= [qexe^\perp + q^\perp e^\perp xe] \xi_0, \\ p^\perp \operatorname{od}_{p\xi_0}(\xi) &= [qe^\perp xe + q^\perp exe^\perp] \xi_0, \\ Sp \operatorname{od}_{p\xi_0}(\xi) &= [qe^\perp x^*e + q^\perp ex^*e^\perp] \xi_0, \\ p^\perp S \operatorname{od}_{p\xi_0}(\xi) &= (qe^\perp + Jq^\perp e^\perp J) [e^\perp x^*e + ex^*e^\perp] \xi_0 \\ &= [qe^\perp x^*e + q^\perp ex^*e^\perp] \xi_0. \end{aligned}$$

Thus it is easy to see that each of the conditions is valid.

We turn to the “if” part. Let p satisfy the conditions of the statement.

Take $x \in \mathcal{M}$ satisfying $x = exe^\perp$. If we use the matrix, x takes the following form.

$$\begin{array}{c} \text{Ran}(e) \\ \text{Ran}(e^\perp) \end{array} \begin{pmatrix} \text{Ran}(e) & \text{Ran}(e^\perp) \\ 0 & X \\ 0 & 0 \end{pmatrix}.$$

Then it holds that $\text{od}_{p\xi_0}(x\xi_0) = x\xi_0$.

By assumption 4, there exists $y \in \mathcal{M}$ such that $px\xi_0 = y\xi_0$. In addition, by assumptions 4a and 4b, we have $eye = e^\perp ye^\perp = 0$, i.e. y has trivial corners. By assumption 4c, it follows $y^2 = 0$. Hence y takes the following form.

$$y = \begin{pmatrix} & & y_1 & 0 & 0 \\ & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & & \\ 0 & y_2 & 0 & & 0 \\ 0 & 0 & 0 & & \end{pmatrix},$$

where we decomposed $\text{Ran}(e)$ and $\text{Ran}(e^\perp)$ as follows.

$$\begin{aligned} \text{Ran}(e) &= \text{Dom}(e^\perp ye) \oplus \text{Ran}(eye^\perp) \oplus (\text{Ran}(e) \ominus \text{Dom}(e^\perp ye) \ominus \text{Ran}(eye^\perp)), \\ \text{Ran}(e^\perp) &= \text{Dom}(eye^\perp) \oplus \text{Ran}(e^\perp ye) \oplus (\text{Ran}(e^\perp) \ominus \text{Dom}(eye^\perp) \ominus \text{Ran}(e^\perp ye)). \end{aligned}$$

Subspaces which appear here are mutually orthogonal because the square of y vanishes.

According to this, we further decompose x .

$$x = \begin{pmatrix} & & x_1 & x_2 & x_3 \\ & 0 & x_4 & x_5 & x_6 \\ & & x_7 & x_8 & x_9 \\ 0 & 0 & 0 & & \\ 0 & 0 & 0 & & 0 \\ 0 & 0 & 0 & & \end{pmatrix}.$$

By assumption 4d, the square of $p^\perp x\xi_0 = (x - y)\xi_0$ must vanish.

$$\begin{aligned} x - y &= \begin{pmatrix} & & x_1 - y_1 & x_2 & x_3 \\ & 0 & x_4 & x_5 & x_6 \\ & & x_7 & x_8 & x_9 \\ 0 & 0 & 0 & & \\ 0 & -y_2 & 0 & & 0 \\ 0 & 0 & 0 & & \end{pmatrix}, \\ (x - y)^2 &= \begin{pmatrix} & & 0 & 0 & 0 \\ & 0 & -y_2x_4 & -y_2x_5 & -y_2x_6 \\ & & 0 & 0 & 0 \\ 0 & -x_2y_2 & 0 & & 0 \\ 0 & -x_5y_2 & 0 & & 0 \\ 0 & -x_8y_2 & 0 & & 0 \end{pmatrix}. \end{aligned}$$

Then it follows that $x_2 = x_4 = x_5 = x_6 = x_8 = 0$.

If we use assumption 4e, then we get

$$px^*\xi_0 = pSx\xi_0 = Sp^\perp x\xi_0 = (x^* - y^*)\xi_0.$$

Applying assumption 4c to $\xi = (x + x^*)\xi_0$, the square of $p(x + x^*)\xi_0 = (y + x^* - y^*)\xi_0$ vanishes.

$$\begin{aligned} y + x^* - y^* &= \begin{pmatrix} & & & & y_1 & 0 & 0 \\ & & & & 0 & -y_2^* & 0 \\ & & & & 0 & 0 & 0 \\ x_1^* - y_1^* & 0 & x_7^* & & & & \\ 0 & y_2 & 0 & & & & \\ x_3^* & 0 & x_9^* & & & & \end{pmatrix}, \\ (y + x^* - y^*)^2 &= \begin{pmatrix} y_1(x_1^* - y_1^*) & 0 & y_1x_7^* & & & & \\ 0 & -y_2^*y_2 & 0 & & & & \\ 0 & 0 & 0 & & & & \\ & & & (x_1^* - y_1^*)y_1 & 0 & 0 & \\ & & & 0 & -y_2y_2^* & 0 & \\ & & & x_3^*y_1 & 0 & 0 & \end{pmatrix}. \end{aligned}$$

Thus it follows that $y_2 = x_3 = x_7 = 0$ and $x_1 = y_1$.

Summing up, for every $x = exe^\perp \in \mathcal{M}$ we have

$$\begin{aligned} x &= \begin{pmatrix} & & & x_1 & 0 & 0 \\ & & & 0 & 0 & 0 \\ & & & 0 & 0 & x_9 \\ 0 & 0 & 0 & & & \\ 0 & 0 & 0 & & & \\ 0 & 0 & 0 & & & \end{pmatrix}, \\ y\xi_0 = px\xi_0 &= \begin{pmatrix} & & & x_1 & 0 & 0 \\ & & & 0 & 0 & 0 \\ & & & 0 & 0 & 0 \\ 0 & 0 & 0 & & & \\ 0 & 0 & 0 & & & \\ 0 & 0 & 0 & & & \end{pmatrix} \xi_0. \end{aligned}$$

The point is that $\text{Dom}(y)$ and $\text{Dom}(x - y)$, $\text{Ran}(y)$ and $\text{Ran}(x - y)$ are mutually orthogonal, respectively.

If we take another element $z = eze^\perp \in \mathcal{M}$ and put $w\xi_0 = pz\xi_0$, then by the same argument we see that $\text{Dom}(w)$ and $\text{Dom}(z - w)$, $\text{Ran}(w)$ and $\text{Ran}(z - w)$ are mutually orthogonal, respectively. In addition, by noting that $w + x - y = e(w + x - y)e^\perp$ and $p(w + x - y)\xi_0 = w\xi_0$, it follows that $\text{Dom}(x - y) \perp \text{Dom}(w)$ and $\text{Ran}(x - y) \perp \text{Ran}(w)$. Similarly it holds that $\text{Dom}(z - w) \perp \text{Dom}(y)$ and $\text{Ran}(z - w) \perp \text{Ran}(y)$. Then let us define f_1 (resp. f_3) to be the projection onto

the supremum of such $\text{Ran}(x - y)$'s (resp. $\text{Dom}(x - y)$'s) where $x = exe^\perp$ runs all the elements of this form in \mathcal{M} and put $f_2 = e - f_1$, $f_4 = e^\perp - f_3$. They are mutually orthogonal projections of \mathcal{M} .

Using them every $x = exe^\perp \in \mathcal{M}$ is decomposed as follows.

$$\begin{array}{cccc} & \text{Ran}(f_1) & \text{Ran}(f_2) & \text{Ran}(f_3) & \text{Ran}(f_4) \\ \begin{array}{l} \text{Ran}(f_1) \\ \text{Ran}(f_2) \\ \text{Ran}(f_3) \\ \text{Ran}(f_4) \end{array} & \left(\begin{array}{cccc} 0 & 0 & x_1 & 0 \\ 0 & 0 & 0 & x_2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right) \end{array}.$$

According to this decomposition, it is easy to see that every $x \in \mathcal{M}$ must have the following form.

$$x = \begin{pmatrix} x_1 & 0 & x_3 & 0 \\ 0 & x_2 & 0 & x_4 \\ x_5 & 0 & x_7 & 0 \\ 0 & x_6 & 0 & x_8 \end{pmatrix}.$$

Put $q = f_1 + f_3$. This is clearly a central projection.

Since p preserves vectors of the set $\{\zeta \mid \zeta \leq p\xi_0 = e\xi_0\}$ by assumption 2, it holds that $p exe\xi_0 = exe\xi_0$ for $x \in \mathcal{M}$. Similarly, by assumption 3, we see $p^\perp e^\perp xe^\perp \xi_0 = e^\perp xe^\perp \xi_0$, hence $p e^\perp xe^\perp \xi_0 = 0$.

Now, letting x be an arbitrary element of \mathcal{M} , p acts on $x\xi_0$ as follows.

$$\begin{aligned} px\xi_0 &= p \begin{pmatrix} x_1 & 0 & x_3 & 0 \\ 0 & x_2 & 0 & x_4 \\ x_5 & 0 & x_7 & 0 \\ 0 & x_6 & 0 & x_8 \end{pmatrix} \xi_0 = \begin{pmatrix} x_1 & 0 & x_3 & 0 \\ 0 & x_2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & x_6 & 0 & 0 \end{pmatrix} \xi_0 \\ &= (qex + q^\perp xe)\xi_0. \end{aligned}$$

Then using the cyclicity of ξ_0 and Lemma 20, we arrive at the conclusion that $p = qe + Jq^\perp eJ$. \square

Corollary 22 *If $\mathcal{M}^\sigma = \mathbb{C}I$, then the conditions in Theorem 21 assure that p is a projection of \mathcal{M} .*

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